

Energy loss of a heavy quark from asymptotically AdS geometries

Christopher P. Herzog

*Department of Physics, University of Washington
Seattle, WA 98195-1560, U.S.A.*

E-mail: herzog@phys.washington.edu

ABSTRACT: We investigate some universal features of AdS/CFT models of heavy quark energy loss. In addition, as a specific example, we examine quark damping in the spinning D3-brane solution dual to $\mathcal{N} = 4$ SU(N) super Yang-Mills at finite temperature and R-charge chemical potential.

KEYWORDS: Heavy Quark Physics, Thermal Field Theory, AdS-CFT Correspondence.

JHEP09(2006)032

Contents

1. Introduction	1
2. The equations of motion	4
3. An analytic, time dependent solution	6
4. Linear analysis	7
4.1 Light quark limit	9
4.2 Dispersion relations	9
5. An example: The R-charged black D3-brane background	10
5.1 Moving quark	11
5.2 Quasinormal modes	14

1. Introduction

We continue the program begun in [1] of using the AdS/CFT correspondence [2–4] to model the energy dissipation of a heavy quark moving through a plasma.¹ Damping of heavy quarks is interesting experimentally for understanding charm and bottom physics at RHIC [10–12]. The traditional field theoretic approach to the problem is perturbative, assuming that the quark interacts weakly with the surrounding plasma via two-body collisions with thermal quarks and gluons and via gluon bremsstrahlung (see [1] for a list of relevant papers). AdS/CFT provides a dual model where one can calculate the energy dissipation at strong coupling, a regime potentially more interesting for RHIC physics where the effective α_s is believed to be of order one.

A direct comparison of AdS/CFT and RHIC data is fraught with difficulty. The essential problem is that AdS/CFT does not provide a dual model of QCD with three flavors. Instead, the original correspondence provides a duality between the maximally supersymmetric $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory and type IIB string theory in a $AdS_5 \times S^5$ background. String theory in curved space is not under good theoretical control, but in

¹Recently, several other papers discussing quark damping from other AdS/CFT perspectives have appeared [5–9].

the limit $N \rightarrow \infty$ and of large 't Hooft coupling $\lambda = g_{YM}^2 N$, the string theory is well approximated by classical supergravity, and strong coupling calculations on the Yang-Mills side reduce to classical calculations in general relativity.

Since the introduction of the original AdS/CFT correspondence, a number of generalizations have been made, some of which are relevant for our discussion. By adding a black hole to the supergravity description, the field theory is raised to a finite temperature [13] dual to the Hawking temperature of the black hole. By introducing black holes spinning in the S^5 direction, the field theory is raised to a finite chemical potential associated to the R-charges of the supersymmetry algebra [14–18]. More recently, [19] argued that adding D7-branes to the geometry was dual to adding flavor hypermultiplets to the gauge theory.

The result of these refinements is a strongly coupled field theory in the large N and λ limit with $\mathcal{N} = 2$ supersymmetry, a field theory with markedly different field content and interactions than three flavor $N = 3$ QCD. Nevertheless, at finite temperature, the theories may not be so different. For example, the pressure divided by the free Stefan-Boltzmann limit (which effectively just counts the number of degrees of freedom) in $\mathcal{N} = 4$ SYM is remarkably close to the corresponding ratio in QCD at temperatures of a few times T_c where it is strongly coupled [20]. The dimensionless ratio of viscosity divided by entropy density equals $1/4\pi$ in $\mathcal{N} = 4$ SYM, as well as in all other theories with gravity duals [21, 22] in the strong 't Hooft coupling limit. And this value, which is lower than any weakly coupled theory or known material substance [23], is in good agreement with hydrodynamic modeling of RHIC collisions [24, 25].

Although one could introduce a chemical potential for heavy quarks, a $U(1)_R$ potential is a better way of capturing effects analogous to a light quark chemical potential. For heavy quarks, introducing a baryon number chemical potential in the AdS/CFT context would mean introducing a macroscopic density of heavy quark baryons. More physically relevant is a situation with a few heavy quark probes moving through a soup with a density of ordinary baryons made up of lighter quarks.

The lesson we draw from the successful comparisons of pressure and viscosity is that we should try to search out dimensionless ratios and also universal features of dual AdS/CFT models. To this end, we take a more general perspective than in [1] where attention was restricted to the dual model of finite temperature $\mathcal{N} = 4$ SYM with one $\mathcal{N} = 2$ hypermultiplet. We consider a general form for the metric of the gravity dual which has a horizon, is asymptotically AdS_{d+1} and preserves Poincare invariance on the boundary. Such a metric includes the finite temperature, zero chemical potential case studied in [1] as well as the finite $U(1)_R$ chemical potential case mentioned above. Such a metric ought also to include a number of relevant deformations of SYM, and other more speculative AdS/CFT correspondences in dimensions $d \neq 4$.

To this $d + 1$ dimensional metric, we assume a flavor brane can be added as in [19]. This flavor brane should fill all of the asymptotically AdS_{d+1} space down to some minimal radius $u = u_0$. There will in general be a nontrivial relation between the radius u_0 and the Lagrangian mass m of the quark. We model our heavy quark as a classical string that stretches from the flavor brane to the horizon of the black hole. The rest mass M_{rest} of the quark is the energy of a straight, motionless string stretching from $u = u_0$ to the horizon

$u = u_h$ and will in general be related in a nontrivial way to both m and u_h . In the limit u_0 becomes large compared to the other scales in the problem,² M_{rest} and m scale linearly with u_0 .

There is a substantial amount of interesting physics in the relation between u_0 , m , and M_{rest} which we will for the most part ignore in this paper. It is known that in the case dual to $\mathcal{N} = 4$ SYM at finite temperature and R-charge chemical potential, there is a first order phase transition as u_0 is lowered toward the horizon. For example, in the case of finite temperature and zero potential, at a value $u_0/u_h \approx 1.02$, the flavor brane jumps to a configuration where it intersects the horizon [26–29]. A similar jump occurs at finite potential [30] and perhaps may happen more generally. There are two lessons to keep in mind. One is that in the limit where the radius of curvature L of AdS is kept large, the string will remain classical down to the phase transition point. The second is that the configuration $u_0 = u_h$ is likely unstable.

Although our approach is more general, the tools we use to measure energy dissipation of a heavy quark are similar to [1]. The goal is to calculate the friction coefficient μ in the equation

$$\frac{dP}{dt} = -\mu P, \tag{1.1}$$

where P is the quark momentum. In section 3, we revisit the analytic solution of a string moving at constant velocity v . Such a string is dual to a quark in an external electric field, and we are able to extract the amount of momentum and energy the field must supply to keep the quark in motion. In section 4, we revisit the linearized, quasinormal mode analysis of the string equation of motion. The quasinormal modes give information about the return to equilibrium of the string after small perturbations, and thus tell us about μ in the small v limit. In section 5, we consider a specific example, the R-charge black hole dual to the SYM field theory at finite temperature and $U(1)_R$ chemical potential.

Saving the details for the body of the paper, we make three interesting observations about our results. The first concerns the small v limit of μ . Both the analytic, constant velocity solution and the quasinormal mode analysis confirm that for all the cases considered

$$\mu M_{\text{kin}} = \frac{(4G_N s)^{2/(d-1)}}{2\pi\alpha'} \tag{1.2}$$

where $1/(2\pi\alpha')$ is the string tension, G_N is Newton’s constant, and s is the entropy density. We have introduced a new mass, M_{kin} , the kinetic mass which enters into the dispersion relation for the quark. In the large u_0 limit, we expect $M_{\text{kin}} \sim M_{\text{rest}}$. In the case of asymptotically AdS_5 geometries, the dual field theory should be a variation of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills. In this case $G_N = \pi L^3/2N^2$ where N is the number of colors in the associated $SU(N)$ field theory. Also, $L^2/\alpha' = \sqrt{\lambda}$ where $\lambda = g_{YM}^2 N$ is the ’t Hooft coupling. We find that in this case, (1.2) becomes

$$\mu M_{\text{kin}} = \sqrt{\lambda} \left(\frac{s^2}{2\pi N^4} \right)^{1/3}. \tag{1.3}$$

²At the very least, $u_0 \gg u_h$, but in general there may be other scales.

The second observation concerns the small u_0 limit. Although the flavor brane with $u_0 \approx u_h$ is probably not stable, and a string stretching from such a flavor brane to the horizon is more quantum than classical, it is nevertheless possible to analyze the quasinormal mode problem formally in this limit:

$$\mu = 2\pi T. \tag{1.4}$$

Moreover, in the R-charge black hole case analyzed in section 5, μ is a monotone decreasing function of u_0 , leading us to speculate that μ is always bounded above by $2\pi T$ in the non-relativistic regime.

The last observation concerns the velocity dependence of μ . In [1], limited evidence supported the claim that the heavy quark obeys a relativistic dispersion relation

$$E(P) = M_{\text{rest}} - M_{\text{kin}} + \sqrt{P^2 + M_{\text{kin}}^2}. \tag{1.5}$$

Moreover, in [1], the friction coefficient μ was velocity independent. Assuming that the same dispersion relation holds in the case of finite chemical potential studied in section 5, we are able to extract μM_{kin} from the constant velocity solution of section 3. Our results indicate that μ has a strong velocity dependence, increasing as v increases. Notably, the perturbative calculations of quark damping also have a nontrivial velocity dependence (see for example [31]).

2. The equations of motion

We assume a metric of the form

$$ds^2 = g_{tt}dt^2 + g_{uu}du^2 + g_{xx}\delta_{ij}dx^i dx^j, \tag{2.1}$$

where $i = 1, 2, \dots, d-1$. As $u \rightarrow \infty$, the metric should approach that of AdS_{d+1} with a radius of curvature L :

$$g_{tt} \rightarrow -L^2 u^2; \quad g_{uu} \rightarrow \frac{L^2}{u^2}; \quad g_{xx} \rightarrow L^2 u^2. \tag{2.2}$$

The space is also assumed to have a horizon at $u = u_h$:

$$g_{tt} = (u - u_h)\partial_u g_{tt}(u_h) + \dots; \quad g^{uu} = (u - u_h)\partial_u g^{uu}(u_h) + \dots. \tag{2.3}$$

The metric component g_{xx} is assumed to be finite at $u = u_h$. Finally, we assume that the metric components g_{tt} , g_{uu} , and g_{xx} depend only on the radial coordinate u . As a shorthand, we will take $\partial_u f = f'$ and $\partial_t f = \dot{f}$.

As discussed in the Introduction, this metric includes as special cases a wide variety of space-times dual, via the AdS/CFT correspondence, to strongly coupled field theories. Some examples are finite temperature $\mathcal{N} = 4$ SU(N) SYM in $d = 4$ dimensions discussed in [1], the same finite temperature SYM at finite R-charge chemical potential to be discussed

in section 5, various relevant deformations of SYM, and other more speculative AdS/CFT correspondences in $d \neq 4$.³

The Hawking temperature of this black hole space-time, dual via the AdS/CFT dictionary to the temperature of the field theory, can be computed by checking that the Euclidean continuation of the metric is regular at $u = u_h$. In this case, we find that

$$T = \frac{\sqrt{-(g_{tt})'(g^{uu})'}}{4\pi} \Big|_{u=u_h}. \quad (2.4)$$

The entropy density of the field theory, proportional to the area of the black hole, is

$$s = \frac{(g_{xx})^{(d-1)/2}}{4G_N} \Big|_{u=u_h}. \quad (2.5)$$

We model a quark in the field theory as a classical string in the dual space-time. We derive the equations of motion for the string from the Nambu-Goto action

$$S = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{-G} \quad (2.6)$$

where G_{ab} is the induced metric on the string world-sheet. We take a static gauge where $t = \tau$, $u = \sigma$, and the string only extends in one direction $x(\sigma, \tau)$. Defining $X = (t, u, x)$ and $U \cdot V = U^\mu V^\nu g_{\mu\nu}$ where $g_{\mu\nu}$ is the space-time metric, we find

$$\begin{aligned} -G &= (\dot{X} \cdot X')^2 - (X')^2 (\dot{X})^2 \\ &= -g_{uu}g_{tt} - g_{xx}g_{tt}(x')^2 - g_{uu}g_{xx}\dot{x}^2. \end{aligned} \quad (2.7)$$

The equation of motion is a partial differential equation:

$$\partial_u \frac{-g_{xx}g_{tt}x'}{\sqrt{-G}} - g_{uu}g_{xx}\partial_t \frac{\dot{x}}{\sqrt{-G}} = 0. \quad (2.8)$$

Recall that the canonical momentum densities associated to the string are

$$\pi_\mu^0 = -\frac{1}{2\pi\alpha'} g_{\mu\nu} \frac{(\dot{X} \cdot X')(X^\nu)' - (X')^2(\dot{X}^\nu)}{\sqrt{-G}}, \quad (2.9)$$

$$\pi_\mu^1 = -\frac{1}{2\pi\alpha'} g_{\mu\nu} \frac{(\dot{X} \cdot X')(\dot{X}^\nu) - (\dot{X})^2(X^\nu)'}{\sqrt{-G}}. \quad (2.10)$$

For our string, these expressions reduce to

$$\begin{pmatrix} \pi_x^0 & \pi_x^1 \\ \pi_u^0 & \pi_u^1 \\ \pi_t^0 & \pi_t^1 \end{pmatrix} = \frac{1}{2\pi\alpha'} \frac{1}{\sqrt{-G}} \begin{pmatrix} g_{xx}g_{uu}\dot{x} & g_{xx}g_{tt}x' \\ -g_{uu}g_{xx}\dot{x}x' & g_{uu}(g_{tt} + g_{xx}\dot{x}^2) \\ g_{tt}(g_{uu} + g_{xx}(x')^2) & -g_{tt}g_{xx}\dot{x}x' \end{pmatrix}. \quad (2.11)$$

³We are assuming any space transverse to the asymptotically AdS_{d+1} geometry decouples from our probe string calculation. Ideally, we would like to know how our string configuration uplifts to ten or eleven dimensional supergravity.

There is a simple time independent solution to (2.8), namely $x = A$ where A is a constant and the string stretches from a D7-brane at $u = u_0$ to the horizon at $u = u_h$. Let's calculate the total energy of such a configuration:

$$E = - \int_{u_h}^{u_0} \pi_t^0 du = \frac{1}{2\pi\alpha'} \int_{u_h}^{u_0} \sqrt{-g_{tt}g_{uu}} du. \quad (2.12)$$

This energy is naturally associated with the rest mass of the quark $E \equiv M_{\text{rest}}$. If we take $u_0 \rightarrow \infty$, there is a divergence associated to this integral. The limit $u_0 \rightarrow u_h$, on the other hand, is finite. Assuming g_{tt} and g_{uu} are well behaved in between, we may conclude that as $u_0 \rightarrow \infty$,

$$M_{\text{rest}} \rightarrow u_0 L^2 / (2\pi\alpha'). \quad (2.13)$$

3. An analytic, time dependent solution

Assuming $\partial_t x = v$ with v a constant, we will find an analytic solution of (2.8) dual to a single quark moving in an electric field \mathcal{E} . The equation of motion reduces in this case to

$$\partial_u \frac{-g_{xx}g_{tt}x'}{\sqrt{-G}} = 0 \quad (3.1)$$

where

$$-G = -g_{uu}g_{tt} - g_{xx}g_{tt}(x')^2 - g_{uu}g_{xx}v^2. \quad (3.2)$$

Integrating once with respect to u , (3.1) transforms into

$$x' = \frac{\sqrt{-G}}{-g_{xx}g_{tt}} C \quad (3.3)$$

where C is the constant of integration. Solving now for x' yields

$$(x')^2 = -\frac{g_{uu}C^2(g_{tt} + g_{xx}v^2)}{g_{xx}g_{tt}(g_{xx}g_{tt} + C^2)}. \quad (3.4)$$

With these results for x' and \dot{x} in hand, we return to the canonical momentum densities (2.11), finding

$$\pi_x^1 = -\frac{1}{2\pi\alpha'} C \quad ; \quad \pi_t^1 = \frac{1}{2\pi\alpha'} v C. \quad (3.5)$$

If we have an open string, then this string will gain energy and momentum through an endpoint u_0 at a rate

$$\begin{aligned} \frac{dE}{dt} &= \pi_t^1|_{u=u_0} = \frac{1}{2\pi\alpha'} C v, \\ \frac{dP}{dt} &= -\pi_x^1|_{u=u_0} = \frac{1}{2\pi\alpha'} C. \end{aligned} \quad (3.6)$$

and lose an equivalent amount of P and E at the other endpoint.⁴

⁴We would like to thank C. Kozcaz, who independently obtained (3.6), for collaboration in the early stages of this project.

Let's specialize to the case where we have a string that stretches from the D-brane at $u = u_0$ to the horizon $u = u_h$. Such a string can be thought of as a single quark moving in an electric field with strength $\pi_x^1 = -\mathcal{E}$. The electric field comes from the U(1) gauge field living on the D-brane and has nothing to do with the SU(N) gauge field of the field theory dual. This \mathcal{E} feeds energy and momentum into the string at a rate given by (3.6) sufficient to keep the string moving at a constant velocity.

In order for the string to stretch from u_0 to u_h , C has to satisfy a special condition. We know generically that g_{tt} has a zero at $u = u_h$ from which we can conclude that for small v , $g_{tt} + g_{xx}v^2$ has a zero for some $u = u_c > u_h$. Thus in order for $(x')^2$ and $\sqrt{-G}$ to be well defined along the length of the string, the factor in the denominator $g_{xx}g_{tt} + C^2$ must have a zero at the same location $u = u_c$. In general an explicit expression for u_c and C may be difficult to find.

Let us assume, for simplicity, that v is small. In that case, we can find an approximate expression for u_c :

$$u_c = u_h - \left. \frac{v^2 g_{xx}}{(g_{tt})'} \right|_{u=u_h} + \mathcal{O}(v^4). \tag{3.7}$$

From this approximation, we find that

$$C = v g_{xx}(u_h) (1 + \mathcal{O}(v^2)) . \tag{3.8}$$

Thus, putting (3.6) and (3.8) together, we find that

$$\frac{dP}{dt} = -\frac{1}{2\pi\alpha'} v g_{xx}(u_h) (1 + \mathcal{O}(v^2)) . \tag{3.9}$$

4. Linear analysis

In this section we analyze small perturbations of a straight string which stretches from $u = u_0$ to $u = u_h$. This analysis allows us to investigate the friction coefficient μ in the non-relativistic limit for any quark rest mass M_{rest} .

Let's look for a solution to (2.8) where $-G \approx -g_{uu}g_{tt}$. Let us also assume a time dependence of the solution that exhibits exponential damping: $\dot{x} = -\mu x$. With these two assumptions, the equations of motion become

$$\frac{\sqrt{-g_{tt}g^{uu}}}{g_{xx}} \partial_u \sqrt{-g_{tt}g^{uu}} g_{xx} x' = \mu^2 x . \tag{4.1}$$

We are interested in solutions x with standard D-brane boundary conditions, i.e. Neumann boundary conditions at a radius $u = u_0$. Because of the absorptive nature of the black hole, we take “out-going” boundary conditions at the horizon $u = u_h$ [32]. To explain “out-going”, consider the solution to (4.1) close to the horizon. Near $u = u_h$, (4.1) takes the approximate form

$$(4\pi T)^2 (u - u_h) \partial_u (u - u_h) x' = \mu^2 x , \tag{4.2}$$

which has the two solutions

$$x = c_1 (u - u_h)^\gamma + c_2 (u - u_h)^{-\gamma} , \tag{4.3}$$

where

$$\gamma = \frac{\mu}{4\pi T}. \tag{4.4}$$

Out-going boundary conditions means we take $c_1 = 0$; with the $e^{-\mu t}$ time dependence, waves travel into but not out of the event horizon.

As a first step in this linear analysis, we make an assumption that will turn out to correspond to studying heavy quarks. In the next section we will consider light quarks, and in section 5.2, we will study the general case for a specific space-time numerically.

We assume that μ is small and accordingly expand our solution as a power series in μ :

$$x = x_0 + \mu^2 x_1 + \dots \tag{4.5}$$

where now our differential equation $\mathcal{L}x = \mu^2 x$ breaks apart into pieces $\mathcal{L}x_0 = 0$ and $\mathcal{L}x_1 = x_0$. The only solution for the leading term is to take $x_0 = A$ where A is a constant. Solving now for x_1 yields

$$x'_1 = \frac{A}{g_{xx}\sqrt{-g_{tt}g^{uu}}} \int_{u_0} \frac{g_{xx}}{\sqrt{-g_{tt}g^{uu}}} du \tag{4.6}$$

where we have taken the lower bound of integration to be u_0 to satisfy the Neumann boundary conditions.

As a final approximation, we will take u_0 to be very large so that near $u = u_0$, the metric components take the asymptotic form (2.2). Having taken this final limit, we can approximately evaluate $x'_1(u)$ for u close to the horizon. The key to the evaluation is the realization that the integral (4.6) will be dominated by its limit behavior near $u = u_h$ and near u_0 . For $u \approx u_h$,

$$x'_1(u) \approx \frac{A}{4\pi T g_{xx}(u_h)(u - u_h)} \left(-L^2 u_0 + \frac{g_{xx}(u_h)}{4\pi T} \ln(u - u_h) \right). \tag{4.7}$$

Matching this result onto our definition of out-going boundary conditions yields a quasinormal mode condition on μ :

$$\mu = \frac{1}{L^2 u_0} g_{xx}(u_h). \tag{4.8}$$

We now use this result for the quasinormal mode to find an expression for the momentum loss. Identifying the endpoint of the string at $u = u_0$ as a quark, the velocity of the quark obeys the differential equation $\dot{v} = -\mu v$. In the large u_0 limit, we have that the mass is approximately $M_{\text{rest}} \sim u_0 L^2 / (2\pi\alpha')$. Putting these two facts together, we find that

$$\frac{dP}{dt} \approx -\frac{u_0 L^2}{2\pi\alpha'} \mu v = -\frac{1}{2\pi\alpha'} v g_{xx}(u_h) \tag{4.9}$$

in perfect agreement with the result (3.9) of the previous section for a slowly moving heavy quark.

4.1 Light quark limit

Having found an analytic expression for μ in the limit $u_0 \rightarrow \infty$, we now investigate the opposite limit $u_0 \rightarrow u_h$. As discussed in the Introduction, this limit corresponds to relatively light quarks. In section 2, we made some assumptions about the near horizon behavior of the metric components. To make progress here, we need to make a few additional assumptions:

$$\sqrt{-g_{tt}g^{uu}} = 4\pi T(u - u_h)(1 + c(u - u_h) + \dots) \quad (4.10)$$

$$g_{xx} = g_{xx}(u_h) + g'_{xx}(u_h)(u - u_h) + \dots, \quad (4.11)$$

where c is a u independent expression that depends on the details of the metric.⁵

Assuming a power series solution to (4.1) of the form

$$x(u) = (u - u_h)^{-\gamma}(1 + A(u - u_h) + \dots), \quad (4.12)$$

we solve for A . The result is that

$$A = \gamma \left(c + \frac{g'_{xx}(u_h)/g_{xx}(u_h)}{1 - 2\gamma} \right). \quad (4.13)$$

Our power series expression for $x(u)$ satisfies the required out-going boundary conditions at the horizon. We also require Neumann boundary conditions at the flavor brane: $x'(u_0) = 0$. Generically in the limit $u_0 \rightarrow u_h$, we expect the first few terms in the power series expansion for $x(u)$ to be dominant. To satisfy Neumann boundary conditions, a sufficient condition is the requirement that $(u_0 - u_h)/(1 - 2\gamma)$ be held fixed in the limit $u_0 \rightarrow u_h$. In this way, there is a possibility that the second term in the power series expansion for $x'(u)$ can cancel the first one at $u = u_0$. But this condition tells us that the friction coefficient in this limit must be

$$\mu = 2\pi T. \quad (4.14)$$

Given that in the limit $u_0 \rightarrow \infty$, μ scales as T/u_0 , it is tempting to speculate that μ is a monotonically decreasing function of u_0 . In the example we study in section 5, this monotone behavior holds. Given such a monotone behavior, it is tempting to go even further and speculate that μ/T is bounded above by 2π for every AdS/CFT model of quark damping.

4.2 Dispersion relations

Continuing our linear analysis, we attempt to establish a relationship between the energy E and momentum P of the string assuming a time dependence of the form $\dot{x} = -\mu x$ and that \dot{x} and x' are small.

Using the equation of motion (4.1), we can rewrite the momentum density as

$$\pi_x^0 = -\frac{\mu}{2\pi\alpha'} \frac{g_{xx}g_{uu}x}{\sqrt{-g_{tt}g_{uu}}} = -\frac{1}{2\pi\alpha'} \frac{1}{\mu} \partial_u \frac{-g_{tt}g_{xx}x'}{\sqrt{-g_{tt}g_{uu}}}. \quad (4.15)$$

⁵Note that there may well be exotic cases where the metric does not have a regular power series expansion near the horizon, $u \approx u_h$.

The total momentum integral can now be evaluated

$$P = \int \pi_x^0 du = -\frac{1}{2\pi\alpha'} \frac{1}{\mu} \sqrt{\frac{-g_{tt}}{g_{uu}}} g_{xx} x' \Big|_{u=u_{min}}^{u=u_0}. \quad (4.16)$$

Because of Neumann boundary conditions at the flavor brane, we know $x'(u_0) = 0$. Ideally, we would like to take $u_{min} = u_h$, but there will be a divergence which we regulate by introducing an infrared cutoff $u_{min} > u_h$.

We evaluate the energy in a similar fashion. Now we keep quadratic terms in the expansion of $\sqrt{-G}$, anticipating a non-relativistic dispersion relation. The energy density takes the form

$$\pi_t^0 = -\frac{1}{2\pi\alpha'} \left[\sqrt{-g_{tt}g_{uu}} + \frac{1}{2} g_{xx} \sqrt{\frac{-g_{tt}}{g_{uu}}} (x')^2 + \frac{1}{2} g_{xx} \sqrt{\frac{g_{uu}}{-g_{tt}}} (\dot{x})^2 \right]. \quad (4.17)$$

Integrating by parts and using the linearized equation of motion yields a simple expression for the energy

$$E = -\int \pi_t^0 du = \frac{1}{2\pi\alpha'} \int_{u_{min}}^{u_0} \sqrt{-g_{tt}g_{uu}} du - \frac{1}{4\pi\alpha'} g_{xx} \sqrt{\frac{-g_{tt}}{g_{uu}}} x x' \Big|_{u=u_{min}} \quad (4.18)$$

where we have used the fact that $x'(u_0) = 0$. Using the fact that close to the horizon

$$x \sim (u - u_h)^{-\mu/4\pi T},$$

and recalling the definition of M_{rest} , we find that

$$E = M_{rest} + \frac{1}{2} \frac{P^2}{M_{kin}}. \quad (4.19)$$

where we have defined a kinetic mass

$$M_{kin} \equiv \frac{g_{xx}(u_h)}{(2\pi\alpha')\mu}. \quad (4.20)$$

In other words, we have found that the quark obeys essentially the usual, non-relativistic dispersion relation for a point particle. The only difference is that the rest mass is different from the kinetic mass.

5. An example: The R-charged black D3-brane background

Consider the following asymptotically AdS_5 metric with horizon [15]:

$$ds^2 = -\mathcal{H}^{-2/3} \frac{(\pi T_0 L)^2}{\rho} f dt^2 + \mathcal{H}^{1/3} \frac{(\pi T_0 L)^2}{\rho} (dx^2 + dy^2 + dz^2) + \mathcal{H}^{1/3} \frac{L^2}{4f\rho^2} d\rho^2; \quad (5.1)$$

where

$$\mathcal{H} = \prod_{i=1}^3 H_i; \quad H_i = 1 + \kappa_i \rho; \quad f = \mathcal{H} - \rho^2 \prod_{i=1}^3 (1 + \kappa_i), \quad (5.2)$$

$$T_0 = \frac{r_+}{\pi L^2}; \quad \kappa_i = \frac{q_i}{r_+^2}, \quad (5.3)$$

and with respect to the radial coordinate u of previous sections $\rho = r_+^2/(u^2 L^4)$. With this change of variables in mind, note that $u_h = r_+/L^2$. The Hawking temperature of the black hole solution is

$$T = \frac{2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1 \kappa_2 \kappa_3}{2\sqrt{(1 + \kappa_1)(1 + \kappa_2)(1 + \kappa_3)}} T_0. \quad (5.4)$$

This gravitational background is dual to $\mathcal{N} = 4$ SU(N) Super Yang-Mills theory with finite chemical potential for the U(1) R-charges. The chemical potentials are related to the κ_i via

$$\phi_i = \pi T_0 \frac{\sqrt{2\kappa_i}}{1 + \kappa_i} \prod_{j=1}^3 (1 + \kappa_j)^{1/2}. \quad (5.5)$$

For convenience, we will express the masses and friction coefficient in terms of the κ_i rather than the ϕ_i .

The black hole provides a model in which to explore the effects of chemical potential on quark damping. We will find two interesting effects. The first is that the friction coefficient is not a monotonic function of the chemical potential. The second is, assuming a relativistic dispersion relation for the quark, that the friction coefficient has nontrivial velocity dependence, unlike the zero chemical potential case studied in [1].

5.1 Moving quark

We begin with a discussion of the analytic, single quark solution discussed in section 3. Formally, from section 3, we know that $dP/dt = -C/2\pi\alpha'$. It is tempting to reorganize this information assuming a relativistic, single particle dispersion relation for the quark

$$P = \frac{M_{\text{kin}} v}{\sqrt{1 - v^2}}. \quad (5.6)$$

In this case, the friction coefficient μ can be expressed in terms of C as

$$\frac{\mu M_{\text{kin}}}{T^2 \sqrt{\lambda}} = \frac{\sqrt{1 - v^2}}{v} \frac{C}{2\pi T^2 L^2}. \quad (5.7)$$

To consider the small velocity limit of this analytic solution, we evaluate g_{xx} on the horizon:

$$\begin{aligned} g_{xx}(u_h) &= (\pi T_0 L)^2 \prod_{i=1}^3 (1 + \kappa_i)^{1/3} \\ &= (2\pi T L)^2 \frac{\prod_{i=1}^3 (1 + \kappa_i)^{4/3}}{(2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1 \kappa_2 \kappa_3)^2}. \end{aligned}$$

Since $C = v g_{xx}(u_h)$, we find that

$$\frac{\mu M_{\text{kin}}}{T^2 \sqrt{\lambda}} = 2\pi \frac{\prod_{i=1}^3 (1 + \kappa_i)^{4/3}}{(2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1 \kappa_2 \kappa_3)^2} \quad (5.8)$$

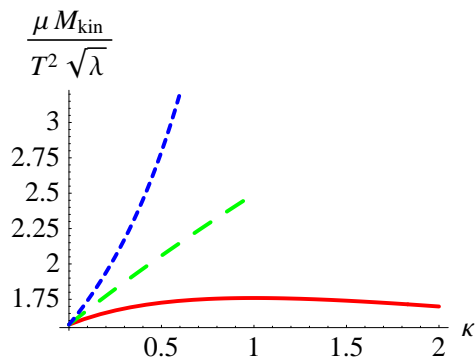


Figure 1: The friction coefficient as a function of κ_i in the small v limit. For the solid red line, $\kappa_1 = \kappa$, $\kappa_2 = \kappa_3 = 0$. For the dashed green line, $\kappa_1 = \kappa_2 = \kappa$, $\kappa_3 = 0$. For the dotted blue line, $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$. The domain of the plot is determined by the region of thermodynamic stability: $2 - \kappa_1 - \kappa_2 - \kappa_3 - \kappa_1\kappa_2\kappa_3 > 0$.

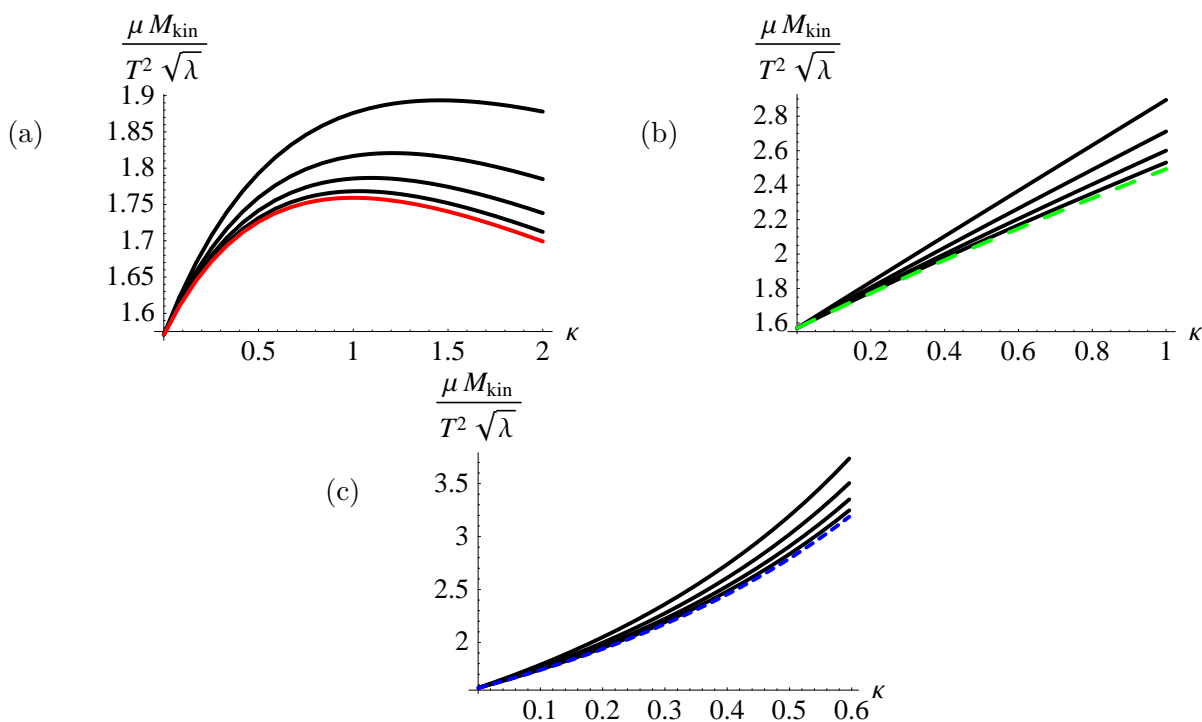


Figure 2: The friction coefficient as a function of κ for different values of v . The bottom most curve is the small v limit. As v increases, the damping increases. Also shown, from bottom to top, are the curves for $v = 0.3, 0.5, 0.7$, and 0.9 : a) $\kappa_1 = \kappa$ and $\kappa_2 = \kappa_3 = 0$, b) $\kappa_1 = \kappa_2 = \kappa$ and $\kappa_3 = 0$, c) $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$.

which is shown plotted in figure 1. Note that the plots are not monotone increasing functions of the chemical potential κ_i . In the case where $\kappa_2 = \kappa_3 = 0$ and only κ_1 is dialed, μM_{kin} reaches a maximum at about $\kappa_1 = 1$.

Although the formulae are messy, one can find explicit expressions for the integration constant C of the analytic, constant velocity solution and hence for the energy and momentum loss. For example, in the case $\kappa_1 = \kappa_2 = \kappa$ and $\kappa_3 = 0$, one finds

$$\frac{\mu M_{\text{kin}}}{T^2 \sqrt{\lambda}} = \frac{\pi}{2} \left(\frac{(1 + \kappa)^2 (1 + \kappa(2 + v^2 \kappa))}{1 + \kappa + \sqrt{1 - v^2 \kappa}} \right)^{1/3}. \quad (5.9)$$

This expression has the small κ expansion

$$\frac{\mu M_{\text{kin}}}{T^2 \sqrt{\lambda}} = \frac{\pi}{2} \left(1 + \frac{1}{3} (3 - \sqrt{1 - v^2}) \kappa + \mathcal{O}(\kappa^2) \right). \quad (5.10)$$

and the small v expansion

$$\frac{\mu M_{\text{kin}}}{T^2 \sqrt{\lambda}} = \frac{\pi}{2} (1 + \kappa)^{2/3} \left(1 + \frac{1}{6} \kappa v^2 + \mathcal{O}(v^4) \right). \quad (5.11)$$

Another simpler case is $\kappa_1 = \kappa$ and $\kappa_2 = \kappa_3 = 0$, for which we find

$$\frac{\mu M_{\text{kin}}}{T^2 \sqrt{\lambda}} = \frac{2^{1/6} 2\pi (1 + \kappa)^{5/3}}{(2 + \kappa)^2 \left(2 + 2\kappa + \kappa^2 - v^2 \kappa^2 + \kappa \sqrt{(1 - v^2)((2 + \kappa)^2 - v^2 \kappa^2)} \right)^{1/6}}. \quad (5.12)$$

This expression has the small κ expansion

$$\frac{\mu M_{\text{kin}}}{T^2 \sqrt{\lambda}} = \frac{\pi}{2} \left(1 + \frac{1}{6} (3 - \sqrt{1 - v^2}) \kappa + \mathcal{O}(\kappa^2) \right), \quad (5.13)$$

and the small v expansion

$$\frac{\mu M_{\text{kin}}}{T^2 \sqrt{\lambda}} = 2\pi \frac{(1 + \kappa)^{4/3}}{(2 + \kappa)^2} \left(1 + \frac{\kappa}{6(2 + \kappa)} v^2 + \mathcal{O}(v^4) \right). \quad (5.14)$$

While the $\kappa_i = 0$ result for μM_{kin} is independent of the velocity, interestingly, nonzero chemical potential introduces a nontrivial dependence of μ on v . As v increases, as is clear from figure 2, μ increases.

Before moving on to an analysis of the quasinormal modes for our string, we consider the relativistic limit $v \rightarrow 1$ of (5.9) and (5.12). Both of these expressions for μM_{kin} approach a finite limit as $v \rightarrow 1$. In the case $\kappa_1 = \kappa_2 = \kappa$ and $\kappa_3 = 0$, we find

$$\frac{\mu M_{\text{kin}}}{T^2 \sqrt{\lambda}} = \frac{\pi}{2} (1 + \kappa) \left(1 - \frac{\kappa \sqrt{1 - v^2}}{3(1 + \kappa)} + \mathcal{O}(1 - v^2) \right) \quad (5.15)$$

while in the case $\kappa_1 = \kappa$ and $\kappa_2 = \kappa_3 = 0$, we get

$$\frac{\mu M_{\text{kin}}}{T^2 \sqrt{\lambda}} = \frac{2\pi (1 + \kappa)^{3/2}}{(2 + \kappa)^2} \left(1 - \frac{\kappa \sqrt{1 - v^2}}{6\sqrt{1 + \kappa}} + \mathcal{O}(1 - v^2) \right). \quad (5.16)$$

Although μM_{kin} is finite in the relativistic limit, the derivative of μ with respect to v diverges at $v = 1$ for nonzero chemical potential.

5.2 Quasinormal modes

We were not able to solve analytically the linearized equation of motion (4.1) for the string in this black hole background. However, we were able to find μ as a function of u_0 numerically. A simple shooting algorithm suffices. At a point close to the horizon $u = (1 + \epsilon)u_h$, we use (4.12) to enforce the out-going boundary conditions. For various values of μ we integrate (4.1) out to the flavor brane $u = u_0$. By refining the choice of μ , we locate the value that satisfies Neumann boundary conditions $x'(u_0) = 0$.

In figure 3, we plot the friction coefficient as a function of M_{rest} for various choices of κ_i . We have introduced $\Delta m = \sqrt{\lambda}T/2$ to plot a dimensionless quantity for M_{rest} . As predicted from the analysis of section 4, $\mu \rightarrow 2\pi T$ in the limit $u_0 \rightarrow u_h$. The plots are also consistent with the prediction that μ scales as $1/u_0$ in the large u_0 limit. In between these two limits, μ is a monotone decreasing function of u_0 , lending credence to our hypothesis that μ is bounded above by $2\pi T$ for small v .

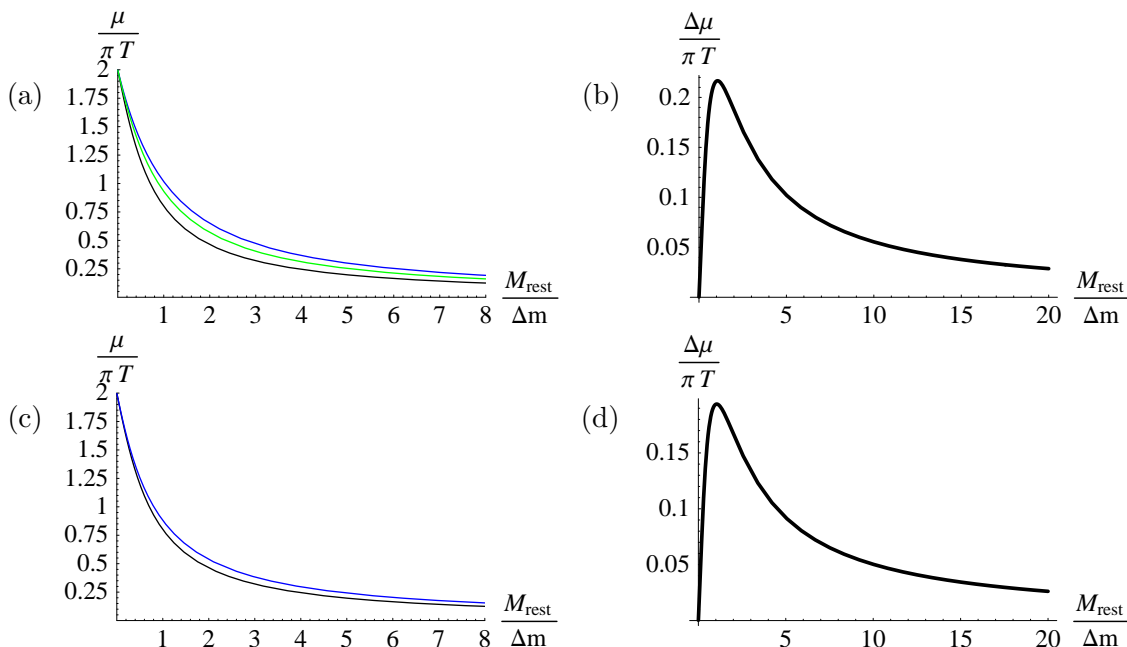


Figure 3: The friction coefficient as a function of M_{rest} for different values of κ . Plots (a) and (b) have $\kappa = \kappa_1 = \kappa_2$ and $\kappa_3 = 0$. Plots (c) and (d) have $\kappa = \kappa_1 = \kappa_2 = \kappa_3$. Plot (a) shows from bottom to top $\kappa = 0, 1/2$, and 1. Plot (c) shows from bottom to top $\kappa = 0$ and $\kappa = 0.596$. Plots (b) and (d) are the difference between the top and bottom curve in (a) and (c) respectively.

Acknowledgments

I would like to thank C. Kozcaz and D. Marolf for useful discussions. Special thanks go to A. Karch and L. Yaffe for their encouragement and comments on the manuscript. This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-96ER40956.

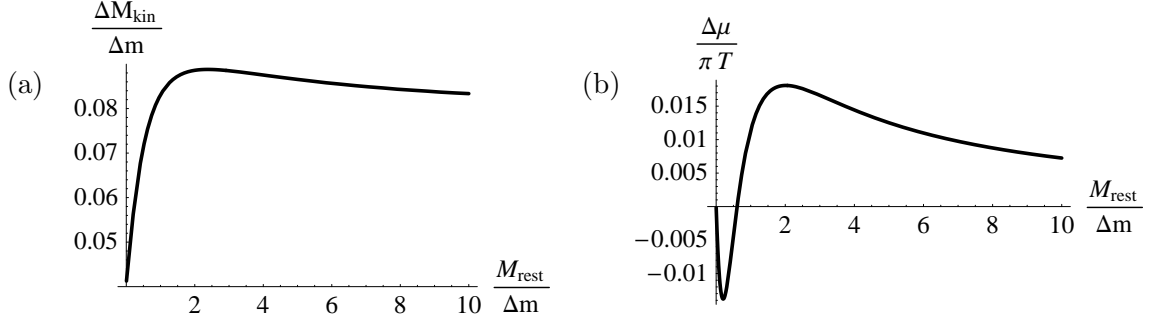


Figure 4: Here $\kappa_2 = \kappa_3 = 0$ and $\kappa_1 = \kappa$. Plot (a) is the difference $M_{\text{kin}}(\kappa=2) - M_{\text{kin}}(\kappa=0)$ as a function of M_{rest} . Plot (b) is the difference in μ for the same choices of κ .

References

- [1] C.P. Herzog, A. Karch, P. Kovtun, C. Kozcaz and L.G. Yaffe, *Energy loss of a heavy quark moving through $N = 4$ supersymmetric Yang-Mills plasma*, *JHEP* **07** (2006) 013 [[hep-th/0605158](#)].
- [2] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [[hep-th/9711200](#)].
- [3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [[hep-th/9802109](#)].
- [4] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)].
- [5] H. Liu, K. Rajagopal and U.A. Wiedemann, *Calculating the jet quenching parameter from AdS/CFT* , [hep-ph/0605178](#).
- [6] J. Casalderrey-Solana and D. Teaney, *Heavy quark diffusion in strongly coupled $N = 4$ Yang-Mills*, [hep-ph/0605199](#).
- [7] S.S. Gubser, *Drag force in AdS/CFT* , [hep-th/0605182](#).
- [8] A. Buchel, *On jet quenching parameters in strongly coupled non-conformal gauge theories*, [hep-th/0605178](#).
- [9] E. Caceres and A. Guijosa, *Drag force in charged $N = 4$ SYM plasma*, [hep-th/0605235](#).
- [10] PHENIX collaboration, S.S. Adler et al., *Nuclear modification of electron spectra and implications for heavy quark energy loss $Au + Au$ collisions at $s(NN)^{1/2} = 200$ GeV*, *Phys. Rev. Lett.* **96** (2006) 032301 [[nucl-ex/0510047](#)].
- [11] STAR collaboration, M. Calderon de la Barca Sanchez et al., *Open charm production from $d + Au$ collisions in STAR*, *Eur. Phys. J.* **C 43** (2005) 187.
- [12] STAR collaboration, A.A.P. Suaide et al., *Charm production in the STAR experiment at RHIC*, *Eur. Phys. J.* **C 43** (2005) 193.
- [13] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [[hep-th/9803131](#)].

- [14] S.S. Gubser, *Thermodynamics of spinning D3-branes*, *Nucl. Phys.* **B 551** (1999) 667 [[hep-th/9810225](#)].
- [15] K. Behrndt, M. Cvetič and W.A. Sabra, *Non-extreme black holes of five dimensional $N = 2$ AdS supergravity*, *Nucl. Phys.* **B 553** (1999) 317 [[hep-th/9810227](#)].
- [16] R.-G. Cai and K.-S. Soh, *Critical behavior in the rotating D-branes*, *Mod. Phys. Lett.* **A 14** (1999) 1895 [[hep-th/9812121](#)].
- [17] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys. Rev.* **D 60** (1999) 064018 [[hep-th/9902170](#)].
- [18] M. Cvetič and S.S. Gubser, *Phases of R-charged black holes, spinning branes and strongly coupled gauge theories*, *JHEP* **04** (1999) 024 [[hep-th/9902195](#)].
- [19] A. Karch and E. Katz, *Adding flavor to AdS/CFT*, *JHEP* **06** (2002) 043 [[hep-th/0205236](#)].
- [20] S.S. Gubser, I.R. Klebanov and A.W. Peet, *Entropy and temperature of black 3-branes*, *Phys. Rev.* **D 54** (1996) 3915 [[hep-th/9602135](#)].
- [21] P. Kovtun, D.T. Son and A.O. Starinets, *Holography and hydrodynamics: diffusion on stretched horizons*, *JHEP* **10** (2003) 064 [[hep-th/0309213](#)].
- [22] A. Buchel, *On universality of stress-energy tensor correlation functions in supergravity*, *Phys. Lett.* **B 609** (2005) 392 [[hep-th/0408095](#)].
- [23] P. Kovtun, D.T. Son and A.O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys. Rev. Lett.* **94** (2005) 111601 [[hep-th/0405231](#)].
- [24] E. Shuryak, *Why does the quark gluon plasma at RHIC behave as a nearly ideal fluid?*, *Prog. Part. Nucl. Phys.* **53** (2004) 273–303 [[hep-ph/0312227](#)].
- [25] E.V. Shuryak, *What RHIC experiments and theory tell us about properties of quark-gluon plasma?*, *Nucl. Phys.* **A 750** (2005) 64 [[hep-ph/0405066](#)].
- [26] J. Babington, J. Erdmenger, N.J. Evans, Z. Guralnik and I. Kirsch, *Chiral symmetry breaking and pions in non-supersymmetric gauge/gravity duals*, *Phys. Rev.* **D 69** (2004) 066007 [[hep-th/0306018](#)].
- [27] A. Karch and A. O’Bannon, *Chiral transition of $N = 4$ super Yang-Mills with flavor on a 3-sphere*, [hep-th/0605120](#).
- [28] D. Mateos, R.C. Myers and R.M. Thomson, *Holographic phase transitions with fundamental matter*, [hep-th/0605046](#).
- [29] T. Albash, V. Filev, C.V. Johnson and A. Kundu, *A topology-changing phase transition and the dynamics of flavour*, [hep-th/0605088](#).
- [30] T. Albash, V. Filev, C.V. Johnson and A. Kundu, *Global currents, phase transitions and chiral symmetry breaking in large- N_c gauge theory*, [hep-th/0605175](#).
- [31] G.D. Moore and D. Teaney, *How much do heavy quarks thermalize in a heavy ion collision?*, *Phys. Rev.* **D 71** (2005) 064904 [[hep-ph/0412346](#)].
- [32] D.T. Son and A.O. Starinets, *Minkowski-space correlators in AdS/CFT correspondence: recipe and applications*, *JHEP* **09** (2002) 042 [[hep-th/0205051](#)].